HOMOCLINIC AND HETEROCLINIC BIFURCATIONS

- Heteroclinic orbits
- Homoclinic orbits
- Homoclinic bifurcations in 2\textsuperscript{nd}- and 3\textsuperscript{rd}-order systems
HETEROCLINIC ORBITS

An orbit (trajectory) started at point $x(0) = x$ is called heteroclinic to the equilibria $\bar{x}'$ and $\bar{x}''$ if $x(t) \to \bar{x}'$ as $t \to +\infty$, and $x(t) \to \bar{x}''$ as $t \to -\infty$. 
Heteroclinic bifurcation: the heteroclinic orbit disappears.
HOMOCLINIC ORBITS

An orbit (trajectory) started at point $x(0) = x$ is called homoclinic to the equilibrium $\bar{x}$ if

$x(t) \to \bar{x}$ as $t \to \pm \infty$. 

In 2nd-order systems, the region inside the homoclinic orbit must be invariant.

If, for example, the homoclinic orbit is attracting (from inside), we observe the so-called “saddle effect”.

An orbit homoclinic to a hyperbolic equilibrium is NOT structurally stable.

⇒ Homoclinic bifurcation
TWO-DIMENSIONAL SYSTEMS (Andronov-Leontovich theorem)

\[ \dot{x}(t) = f(x(t), p), \quad x \in \mathbb{R}^2 \]

The system has a saddle \( \bar{x} \) and a homoclinic orbit when \( p = \bar{p} \).

What happens at \( p \) close to \( \bar{p} \)?

Generically (under nondegeneracy conditions) the homoclinic orbit disappears, and a limit cycle exists on one side of the bifurcation.

The stability of the limit cycle depends upon the sign of the saddle quantity \( \sigma = \lambda_1 + \lambda_2 \).
Case I: $\sigma < 0$ - stable limit cycle

The homoclinic bifurcation can be seen as the collision of the saddle with a stable limit cycle.

As $p \to \bar{p}$ from above, the limit cycle becomes more and more “angled” (similar to the homoclinic orbit) and its period $\tau \to \infty$. 
Example: tritrophic food chain

\(x_1(t), x_2(t)\) = biomass of prey and predator  \(x_3\) = biomass of super-predator (constant)

\[\begin{align*}
\dot{x}_1 &= rx_1 \left(1 - \frac{x_1}{k}\right) - \frac{ax_1x_2}{b + x_1} \\
\dot{x}_2 &= -mx_2 + e \frac{ax_1x_2}{b + x_1} - c \frac{x_2}{d + x_2} x_3
\end{align*}\]
**Case II:** \( \sigma > 0 \) - unstable limit cycle

The homoclinic bifurcation can be seen as the collision of the saddle with an unstable limit cycle.
**Example:** adaptive control system (*Salam and Bai, 1986*)

\[
\dot{x}_1 = -x_1 + x_1 x_2 + d
\]

\[
\dot{x}_2 = -0.5 x_2 + x_1^2 - 1.25
\]

\(x_1\) = output error,
\(x_2\) = parameter error,
\(d\) = disturbance (constant)
THREE-DIMENSIONAL SYSTEMS (Shilnikov theorems)

\[ \dot{x}(t) = f(x(t), p), \quad x \in \mathbb{R}^3 \]

The system has a saddle \( \bar{x} \) and a homoclinic orbit when \( p = \bar{p} \).

- \( \bar{x} \) has all real eigenvalues (saddle)
- \( \bar{x} \) has complex eigenvalues (saddle-focus)

\[ \sigma = \lambda_1 + \lambda_2 \]

saddle quantity

\[ \sigma = \lambda_1 + \text{Re}(\lambda_2) \]

saddle quantity
The number and stability of the limit cycles generated by the homoclinic bifurcation depend upon the type of saddle and upon the saddle quantity.

<table>
<thead>
<tr>
<th>$\bar{x}$ is a saddle</th>
<th>$\bar{x}$ is a saddle-focus</th>
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<tbody>
<tr>
<td>$\sigma &lt; 0$</td>
<td>1 asymptotically stable cycle</td>
</tr>
<tr>
<td>$\sigma &gt; 0$</td>
<td>1 saddle cycle</td>
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</table>
Case I: $\sigma < 0$ - $x$ is a saddle

Case II: $\sigma < 0$ - $x$ is a saddle-focus
Case III: $\sigma > 0$ - $\vec{x}$ is a saddle

"non-twisted":

![Diagram showing non-twisted and twisted cases]
Case IV: $\sigma > 0$ - $\bar{\sigma}$ is a saddle-focus

The system has an infinite number of saddle limit cycles in the neighbourhood of the homoclinic orbit for all sufficiently small $|p - \bar{p}|$.

"Shilnikov chaos": a chaotic attractor may exist in such scenario, with geometry similar to that of the disappeared homoclinic orbit.

Example: a model of power generation and distribution
To complete the picture, there are also asymptotically stable or repelling limit cycles, undergoing an infinite number of saddle-nodes (t) and flip (f) bifurcations.

For a sequence of values of \( p \rightarrow \bar{p} \), there also exist “2-rotation” (and even “higher-rotation”) homoclinic orbits.